Assignment 6.

This homework is due *Thursday*, October 9.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 5.

1. Quick reminder

Measurable sets form a σ-algebra $\mathcal{M}$. The Lebesgue measure is a function $m : \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined as $m(A) = m^*(A)$.

The Lebesgue measure m has the following properties:

- $m(I) = \ell(I)$ for every interval I.
- m is translation invariant: for any $A \in \mathcal{M}$, for any $y \in \mathbb{R}$,
$$m(A + y) = m(A).$$
- m is countably additive, i.e. for measurable disjoint sets $\{A_k\}$,
$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m\left(A_k\right).$$

2. Exercises

(1) (2.4.17) Show that the set E is measurable if and only if for each $\varepsilon > 0$, there is a closed set F and open set O for which $F \subseteq E \subseteq O$ and $m^*(O \setminus F) < \varepsilon$.
(*Hint:* Use outer approximation of E by open sets and inner approximation of E by closed sets.)

(2) ($\sim$2.4.18) Let E have finite outer measure. Show that E is measurable if and only if there is an F_σ set F and a G_σ set G such that
$$F \subseteq E \subseteq G \text{ and } m^*(F) = m^*(E) = m^*(G).$$
(Terminology: a set that is a countable union of closed sets is called an F_σ set. A set that is a countable intersection of open sets is called a G_σ set.)

(3) (Theorem 2.4.12) Let E be a measurable set of finite measure. Then for each $\varepsilon > 0$, there is a finite disjoint collection of open intervals $I_1, \dots, I_n$ for which if $\mathcal{O} = \bigcup_{k=1}^{n} I_k$, then
$$m^*(E \setminus \mathcal{O}) + m^*(\mathcal{O} \setminus E) < \varepsilon.$$
(*Hint:* Get a countable collection $\{I_k\}$ that $\varepsilon/2$-approximates E. Since the series $\sum m(I_k)$ converges, there is a tail with $\sum_{n=k}^{\infty} m(I_k) < \varepsilon/2$.)

— see next page —

(4) (a) (Continuity of m from below; Theorem 2.5.15i) Let $A_1 \subseteq A_2 \subseteq \dots$ be a countable collection of measurable sets. Show that

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k).$$

(*Hint:* Switch to disjoint sets. Then limit becomes a sum of series.)

(b) (Continuity of m from above; Theorem 2.5.15ii) Let $B_1 \supseteq B_2 \supseteq \dots$ be a countable collection of measurable sets and $m(B_1) < \infty$. Show that

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k).$$

(*Hint:* Complement of intersection is union of complements.)

(c) (2.6.25) Show that the assumption $m(B_1) < \infty$ above is necessary.

(5) (2.7.39) Let F be the subset of $[0,1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at nth deletion stage has length $\alpha/3^n$ with $0 < \alpha < 1$ (rather than $1/3^n$). Show that F is a closed set, $[0,1] \setminus F$ is dense in $[0,1]$, and $m(F) = 1 - \alpha$. Such set F is called a generalized Cantor set.

(Reminder: a set E is *dense* in $[0,1]$ if any open interval in $[0,1]$ contains a point from E.)

(6) (2.7.40) Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (*Hint:* Consider the complement of generalized Cantor set.)

(Reminder: for a set $A \in \mathbb{R}$, $x \in \mathbb{R}$ is a *boundary point* of A if for every $\varepsilon > 0$, interval $(x-\varepsilon, x+\varepsilon)$ contains a point from A and from $\mathbb{R} \setminus A$. *Boundary* of a set A is the set of all its boundary points.)

(7) (2.7.44+) A subset A of $\mathbb{R}$ is said to be *nowhere dense* in $\mathbb{R}$ provided that every open set O has an open subset that is disjoint from A. Show that the Cantor set and the generalized Cantor set are nowhere dense in $\mathbb{R}$.
Comment. Hence there are nowhere dense sets of positive measure.